Propositional Logic

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1 Natural Deduction Proofs

There are multiple ways to prove a sequent using ND proofs (in fact it can be trivially proved by construction that there exist infinite ND proofs for any sequent - keep using $\neg\neg$ i and $\neg\neg$ e). For some problems, especially in 1.3 natural language explanation of the proof strategy have been given, but these are just one of the possible proof strategies. Lastly, in some problems, for brevity, Modus Tollens and Modus Ponens, $\neg i$ (PbC), etc. are used as inference rules, but these can be skipped and derived as sub-proofs using more elementary inference rules.

1.1 Natural Deduction - 1

1.1.1 english_summer $\wedge$green_top $\wedge$ fast_bowling, indian_batsmen_fail $\vdash$ english_summer $\wedge$ fast_bowling $\wedge$ indian_batsmen_fail

Line	Deduction	Explanation
1	english_summer $\wedge$ green_top $\wedge$ fast_bowling	Premise
2	english_summer	$\wedge$e1 1
3	fast_bowling	$\wedge$e2 1
4	indian_batsmen_fail	Premise
5	english_summer $\wedge$ fast_bowling	$\wedge$i 2, 3
6	english_summer $\wedge$ fast_bowling $\wedge$ indian_batsmen_fail	$\wedge$i 5, 4

$\therefore$ english_summer $\wedge$ green_top $\wedge$ fast_bowling, indian_batsmen_fail $\vdash$ english_summer $\wedge$ fast_bowling $\wedge$ indian_batsmen_fail

1.1.2 Prove the following sequent : $\mathbf{p \to q, q \to r \vdash p \to r}$

Line	Deduction	Explanation
1	p	Assumption
2	$p \to q$	Premise
3	q	$\to$ e $2, 1$
4	$q \to r$	Premise
5	r	$\to$ e $4, 3$
6	$p \to r$	$\to$ i $1 - 5$

$\therefore p \to$ q, q $\to$ r $\vdash$ p $\to$ r

1.1.3 Using the above proof, prove $\vdash \mathbf{(p \to q) \to ((q \to r) \to (p \to r))}$

Line	Deduction	Explanation
1	$p \to q$	Assumption
2	$q \to r$	Assumption
3	p	Assumption
4	q	$\to$e 1,3
5	r	$\to$e 2,4
6	$p \to r$	$\to$i 3-5
7	$(q \to r) \to (p \to r)$	$\to$i 2-6
8	$(p \to q) \to ((q \to r) \to (p \to r))$	$\to$i 1-7

$\therefore \quad \vdash (p \to q) \to ((q \to r) \to (p \to r))$

1.2 Natural Deduction - 2

1.2.1 Prove i_am_god $\to$ happy, i_am_god $\to \neg$happy $\vdash \neg$i_am_god

Line	Deduction	Explanation
1	*i_am_god*	Assumption
2	*i_am_god* $\to$ *happy*	Premise
3	*happy*	$\to$ e $2, 1$
4	*i_am_god* $\to \neg$*happy*	Premise
5	$\neg$*happy*	$\to$ e $4, 1$
6	$\bot$	$\bot$i $3, 5$
7	$\neg$*i_am_god*	$\neg$i $1 - 6$

$\therefore$ i_am_god $\to$ happy, i_am_god $\to \neg$happy $\vdash \neg$i_am_god

1.2.2 Prove $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$

Line	Deduction	Explanation
1	$(p \wedge q) \vee (p \wedge r)$	Premise
2	$p \wedge q$	Assumption
3	p	$\wedge e_1\, 2$
4	q	$\wedge e_2\, 2$
5	$q \vee r$	$\vee i\, 4$
6	$p \wedge (q \vee r)$	$\wedge i\, 3, 5$
7	$p \wedge r$	Assumption
8	p	$\wedge e_1\, 7$
9	r	$\wedge e_2\, 7$
10	$q \vee r$	$\vee i\, 9$
11	$p \wedge (q \vee r)$	$\wedge i\, 8, 10$
12	$p \wedge (q \vee r)$	$\vee e\, 1, 2-6, 7-11$

$\therefore (p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$

1.3 Natural Deduction - 3

1.3.1 Proof:

$$\neg\textbf{rains} \vee \textbf{wet\_road} \vdash \textbf{rains} \rightarrow \textbf{wet\_road}$$

Line	Deduction	Explanation
1	¬rains	Assumption
2	rains	Assumption
3	⊥	⊥i 1,2
4	wet_road	⊥e 3
5	rains → wet_road	→i 2-4
6	wet_road	Assumption
7	rains	Assumption
8	wet_road	Repetition of Line 6
9	rains → wet_road	→i 7-8
10	rains → wet_road	∨ e 1-5, 6-9

∴ ¬rains ∨ wet_road ⊢ rains → wet_road.

When the LHS contains p ∨ q the simple proof strategy is to prove the RHS for both p and q

1.3.2 Proof:

$$\neg p \vee q \vdash p \rightarrow q$$

Line	Deduction	Explanation
1	$\neg p$	Assumption
2	p	Assumption
3	$\bot$	$\bot$i 1,2
4	q	$\bot$e 3
5	$p \to q$	$\to$i 2-4
6	q	Assumption
7	p	Assumption
8	q	6
9	$p \to q$	$\to$i 7-8
10	$p \to q$	$\vee$e 1-5, 6-9

$$\therefore \neg p \vee q \vdash p \to q.$$

1.3.3 Proof:

$$q \to (p \to r), \neg r, q \vdash \neg p$$

Line	Deduction	Explanation
1	$q \to (p \to r)$	Premise
2	$\neg r$	Premise
3	q	Premise
4	$p \to r$	Modus Ponens (1, 3)
5	p	Assumption
6	r	Modus Ponens (4, 5)
7	$\bot$	$\bot$i 2,6
8	$\neg p$	$\neg$ i 5-7

$$\therefore q \to (p \to r), \neg r, q \vdash \neg p.$$

Try to do this without Modus Ponens (which basically means you need to do a sub proof of MP after line 5)

1.3.4 Proof:

$$p \to q \vdash \neg p \vee q$$

Line	Deduction	Explanation
1	$p \to q$	Premise
2	$\neg(\neg p \vee q)$	Assumption
3	$\neg p$	Assumption
4	$\neg p \vee q$	$\vee$ i 3
5	$\perp$	$\perp$ i 2, 4
6	$\neg(\neg p)$	$\neg$i 3-5
7	p	$\neg\neg$e 6
8	q	Assumption
9	$\neg p \vee q$	$\vee$ i 8
10	$\perp$	$\perp$ i 2, 9
11	$(\neg q)$	$\neg$i 8-10
12	$p \wedge \neg q$	$\wedge$i 7, 11
13	$\neg p$	Modus Tollens 1, 11
14	$\neg p \vee q$	$\vee$ i 13
15	$\perp$	$\perp$i 2, 14
16	$\neg\neg(\neg p \vee q)$	$\neg$i 2-15
17	$\neg p \vee q$	$\neg\neg$e 16

$$\therefore p \to q \vdash \neg p \vee q.$$

The main challenge of this problem is to do it without using LEM. The proof strategy is simple: Assume the negation of the RHS $\to$ do a subproof of DeMorgan's law to distribute the negation over the RHS $\to$ generate a contradiction $\to$ negate the assumption. Notice that lines 2-12 are just there to prove DeMorgan's law.

1.3.5 Proof:

$$\vdash p \vee \neg p$$

Line	Deduction	Explanation
1	$\neg(p \vee \neg p)$	Assumption
2	p	Assumption
3	$p \vee \neg p$	$\vee$i 2
4	$\perp$	$\perp$ i 1,3
5	$\neg p$	$\neg$ i 2
6	$p \vee \neg p$	$\vee$i 5
7	$\perp$	$\perp$ i 1,6
8	$\neg\neg(p \vee \neg p)$	$\neg$i 1, 7
9	$p \vee \neg p$	$\neg\neg$ e 8

$$\therefore \vdash p \vee \neg p$$

1.3.6 Proof:

$$\neg(p \wedge q) \vdash \neg p \vee \neg q$$

Line	Deduction	Explanation
1	$\neg(p \wedge q)$	Premise
2	$\neg(\neg p \vee \neg q)$	Assumption
3	p	Assumption
4	q	Assumption
5	$p \wedge q$	$\wedge$i 3, 4
6	$\perp$	$\perp$i 1, 5
7	$\neg q$	$\neg$ i 4–6
8	$\neg p \vee \neg q$	$\vee$ i 7
9	$\perp$	$\perp$i 2, 8
10	$\neg\neg(\neg p \vee \neg q)$	$\neg$i 2-9
11	$\neg p \vee \neg q$	$\neg\neg$e 10

$$\therefore \neg(p \wedge q) \vdash \neg p \vee \neg q.$$

Notice how this is also just a proof of DeMorgan's law, which is already saw as a subproof in 1.3.4. Try to prove DeMorgan's law for distributing a $\neg$ over p$\vee$q.

1.3.7 Proof:

$$p \to q, r \to s \vdash p \vee r \to q \vee s$$

Line	Deduction	Explanation
1	$p \to q$	Premise
2	$r \to s$	Premise
3	$p \vee r$	Assumption
4	p	Assume
5	q	Modus Ponens 1, 4
6	$q \vee s$	$\vee$i 5
7	r	Assumption
8	s	Modus Ponens 2, 7
9	$q \vee s$	$\vee$i 8
10	$q \vee s$	$\vee$ e 3, 4–6, 7–9
11	$p \vee r \to q \vee s$	$\to$i 1-10

$$\therefore p \to q, r \to s \vdash p \vee r \to q \vee s.$$

1.3.8 Proof:

$$(p \to r) \wedge (q \to r) \vdash (p \wedge q) \to r$$

Line	**Deduction**	**Explanation**
1	$p \to r$	$\wedge$e1 premise
2	$q \to r$	$\wedge$e2 premise
3	$p \wedge q$	Assumption
4	p	$\wedge$e1 3
5	r	Modus Ponens on 1 and 4
9	$(p \wedge q) \to r$	$\to$i 3-5

$$\therefore (p \to r) \wedge (q \to r) \vdash (p \wedge q) \to r.$$